Physics of Finance: Gauge Modelling in Non-equilibrium Pricing
by Kirill Ilinski
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The physical world works something like this: Every point in space-time has its own copy of a vector space. (This is a fiber bundle; the fibers are the vector spaces.) The vectors in these internal spaces, spread out over real space, are called the matter field. There is a rule which specifies how to compare vectors attached to different points — how to do parallel transport. The rule is called a connection, or a gauge field. If, when you move along a closed path, the vector you start with isn't the same as the one you end with, then the gauge field is curved. The bigger the difference between the vectors, the larger the curvature of the connection. Both fields, matter and gauge, change over time, obeying a principle of least action.

The world doesn't care how we measure things. In particular, gauge transformations — rotations of the internal vector spaces — are physically irrelevant. We can rotate the internal space of each point however we like, so long as the rotations are continuous from point to point. The gauge field changes along with the matter fields, so that vectors which were parallel before stay parallel. But the action, and so the motions dictated by the action, must be gauge-invariant. This constrains how the action can depend on the matter and gauge fields. If we choose the simplest gauge-invariant action, it says, in effect, that matter moves along trajectories of minimal curvature, and that the connection grows curved near masses. The gauge field guides the matter field, and the matter field puts curvature in the gauge field; curvature is force.

Each of the fundamental forces of nature (gravity, electromagnetism, the weak and strong nuclear forces) corresponds to a different gauge field, the connection for a fiber bundle with a different internal vector space. Fundamental physics is about fiber bundles, connections, gauge invariance, and statistical fluctuations about the least-action trajectory. Getting a degree in mathematical physics is about learning to do calculations about fiber bundles, connections, gauge invariance and statistical fluctuations[1]. Now, Kirill Ilinski has a degree in mathematical physics ....

Think, he says, about what we do when we calculate net present values. We employ a rule that lets us compare the value of commodity bundles at different times, by specifying equivalent values at a common time. Commodity bundles are vectors in commodity space; calculating net present value is engaging in parallel transport of these vectors according to a connection. Prices let us convert from one commodity to another — they are parts of a connection across a "space" where each different commodity is its own point. To really compare the value of commodity bundles, we need a connection which includes prices and discounting as its spatial and temporal components. Suppose this connection

[1] Ian Lawrie's *A Unified Grand Tour of Theoretical Physics* (IOP, 1990) is an excellent exposition of this approach.

has non-zero curvature. Then we can make a series of exchanges and end up with a commodity bundle whose value is not the same as the one we started with — i.e., we can make money by trading at going prices. Curvature is excess returns; curved paths are arbitrage opportunities. Gauge transformations are changes in the units we use to measure commodities — going from denominating hog bellies by the gross to denominating by the hundred, for instance, or counting money in dollars or pennies. These gauge changes are dilations, not the rotations of physics, but the same idea applies.

So far so good; we've been speaking fiber bundles and connections without knowing it. What is Ilinski giving us, besides new math for what we've done all along? Three things. First, there should be coupling between the "matter" field (the commodities) and the "gauge" field (prices and discount rates). In particular, exploiting arbitrage opportunities should tend to make them go away — matter should redistribute itself so as to even out prices. Second, the dynamics should, just as in physics, be governed by the principle of least action. Here this corresponds in part to utility maximization on the part of traders. Third, the form of the action should be gauge-invariant. But gauge invariance tightly limits the form of the action; there are only a few possibilities which are invariant and simple.

These requirements are not enough to fix the form of the action, and so the dynamics. Ilinski assumes that the connections are stochastic functions of time — that changes in prices are at least partially random. Similarly, the response of the matter fields to the gauge fields — how traders respond to prices — is also stochastic. Traders will generally try to make money rather than lose it, and to make more money rather than less, but their decisions will be probabilistic, and sometimes they'll chose wrong. (He favors the "logit" decision rule, which has some psychological and econometric support, and perhaps more importantly looks very like the Boltzmann factor from statistical physics.) Putting these together, he formulates an action that lets him calculate the price dynamics of asset markets, first with just a single trader, and then with a fluctuating number of interacting traders. (Handling a variable number of traders needs some tricks from field theory, explained in an appendix.) The results of this model match empirical data respectably, at least as well as the results of phenomenological models like GARCH or truncated Lévy flights — and unlike them, this model gives an underlying mechanism. It even explains why pieces of technical analysis that look little better than voodoo can work over short time horizons, while the long run looks comfortingly close to the efficient market hypothesis.

Having built up this model, Ilinski goes on to apply it, in detail, to pricing assets, especially derivatives. It is notable — and he duly notes it — that he needs to make *no* assumption of economic equilibrium, or even of near-equilibrium, though he can reproduce equilibrium results (e.g., Black-Scholes) in the appropriate limits. This material is not just an addendum (it runs nearly a hundred pages), but is too detailed for discussion here.

Is any of this believable? The fiber-bundle/connection formalism certainly is; once someone's pointed it out, it's obvious that prices and discount rates are components of a connection. Nor is it particularly contentious that people respond to prices (that's what they're for), and that trades change prices (that's supply and demand). So it should certainly be possible to formulate market dynamics in terms of the "matter" (commodity) and "gauge" (price) fields. And this isn't peculiar to finance — it should work for markets generally, even with production and consumption.

That a least-action principle should govern the dynamics is less obvious. Ilinski gets one by assuming traders are boundedly rational in a way that's mathematically tractable, and not *outrageous* empirically, but vastly simpler than the real cognitive mechanisms underlying choice and action. How sensitive are his results to the exact form of the traders' decision rules? Are behaviorally-realistic dynamics action-minimizing? Ilinski's traders are (imperfect) utility-maximizers, but they're not *strategic* actors — they're equally oblivious to what they do to others and what others do to them. Is there *any* way of capturing strategic interaction in a minimum-action principle?[2] Even if we give Ilinski an action principle, we don't have to give him gauge invariance. That symmetry makes sense in physics, because our measurement conventions really don't make any difference to Nature. But gauge symmetry is definitely broken in real markets, in many ways — psychological effects, certain kinds of transaction costs, etc. Ilinski knows this, and wants to handle those effects as small perturbations — but are they? All these matters need more investigation.

Ilinski's book is well-written and well-produced. The math is developed essentially from scratch, and very skillfully — his exposition of fiber bundles, path integrals and field-theory methods is one of the best I've read. The writing is clear, and the English perfectly fine (which isn't true of some of his papers). And the models themselves have many virtues: elegant math, neat results, and good agreement with data, and not just by the standards of econophysics. Still, as Chairman Mao said, "practice is the only test of truth", and we don't, yet, know well these models work in financial practice. If this book finds the readers it deserves, we'll soon learn.

[2]Cf. Herbert Gintis, *Game Theory Evolving* (Princeton U.P., 2000).